practice

How would you find:

- $$\int \frac{\mathrm{d}x}{e^x + e^{-x}}$$
- $$\int \frac{e^{2x}}{e^x - 1}\,\mathrm{d}x$$
- $$\int \frac{e^{2x}}{\sqrt{1 - e^x}}\,\mathrm{d}x$$

1. Let u=e^x. Then dx=du/u.
   Ans = arctan (e^x) + C

2. Let u=e^x. Becomes u du / (u-1),
   divide to get u/(u-1)=1+1/(u-1)
   Ans = e^x + ln | e^x - 1 | + C

3. integration by parts, u=e^x
   dv=e^x (1-e^x)^(-1) dx
   Ans = -sqrt(1-e^x) * (4/3 + 2e^x / 3) + C
   (This is Leithold 7.1.20)

today:
§ 7.8 - improper integrals

friday:
last drop day
webwork 5 due @ 11:55 pm
mslc webwork 5 workshop in SEL 040 @ 12:30, 1:30, 2:30, 3:30, 4:30

monday:
webwork extra credit ii help session in EA 265 @ 5:30

tuesday:
§ 8.1 - arc length

thursday, 12 november:
§ 8.2 - surface area
quiz iv: §§ 4.4, 7.8
homework 6 due (4.4.28, 4.4.40, 4.4.58, 7.8.26, 7.8.36, 7.8.40)

monday, 16 november:
webwork extra credit ii help session in EA 265 @ 5:30

monday, 23 november:
webwork extra credit ii due @ 6:00 am

Extra Credit II due date has been moved to Monday, 23 November (this gives you an extra week than before.)

The extra week is the week of midterm iii.

review of l'Hôpital

evaluate

$$\lim_{x \to 1/2^-} \frac{\ln(1-2x)}{\tan(\pi\, x)}$$

This is a - ∞ / ∞, so use l'Hôpital.

Answer=0.

(Leithold 7.8.17)

Ask if there are any other l'Hôpital questions.

review of l'Hôpital

evaluate

$$\lim_{x \to -\infty} x^2\, e^x$$

Answer=0.

This is Stewart's 4.4.38.

review of l'Hôpital

evaluate

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$$

Get a common denominator. l'Hôpital. multiply top and bottom by x. l'Hôpital.

Answer=1/2

This is Stewart 4.4.48.

review of l'Hôpital

evaluate

$$\lim_{x\to 0}(1-2x)^{1/x}$$

Let y=(1-2x)^(1/x), find limit of ln(y) using l'Hôpital. Limit of ln(y) is -2, so the answer is exp(-2).

This is Stewart's 4.4.53.

improper integrals

Here a is a number. In particular, a is not negative infinity.

If f is continuous for all $x \geq a$, we define the improper integral with infinite upper limit

$$\int_a^\infty f(x)\,\mathrm{d}x := \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x$$

provided this limit exists.

Improper integrals are integrals where infinity comes up in some way.

This presentation follows Leithold §§ 7.9 and 7.10.

If the limit does not exist, the integral is divergent. If it does, then convergent.

improper integrals

If f is continuous for all $x \leq b$, we define the improper integral with infinite lower limit

$$\int_{-\infty}^b f(x)\,\mathrm{d}x := \lim_{a\to-\infty} \int_a^b f(x)\,\mathrm{d}x$$

provided this limit exists.

b is a number. b is NOT positive infinity.

Integrals that are improper because of having bounds of ∞ or -∞ are called improper integrals of type 1.

example

evaluate

$$\int_0^\infty x\, e^{-x^2}\, \mathrm{d}x$$

Answer = 1/2

example

evaluate

$$\int_0^\infty \sin x\, \mathrm{d}x$$

This is a divergent integral. No limit exists.

example

evaluate

$$\int_1^\infty \frac{\mathrm{d}x}{x}$$

This is a divergent integral. The limit is ∞.

Fact: If instead the denominator was x^p, would be convergent if p>1, divergent if p≤1.

example

What is the volume of the figure obtained by revolving the region under $f(x)=1/x$, $x\geq 1$ about the x-axis?

We just saw that the area was unbounded, but the volume is finite.

Answer = π

improper integrals

If f is continuous for all x and c is any real number, we define the improper integral with both lower and upper infinite limits

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x := \lim_{a\to-\infty}\int_{a}^{c} f(x)\,\mathrm{d}x + \lim_{b\to\infty}\int_{c}^{b} f(x)\,\mathrm{d}x$$

provided this limit exists.

It is left as an exercise for the reader to show that the answer is the same regardless of the choice of c.

In particular, we do NOT just take one limit. We must split it up, so that we always have at least one finite bound.

example

evaluate

$$\int_{-\infty}^{\infty} \frac{4\,x}{(x^2+1)^3}\,\mathrm{d}x$$

Split at 0 (say). Integral from 0 to ∞ is 1, from -∞ to 0 is -1, so whole integral is 0.

example

evaluate

$$\int_{-\infty}^{\infty} \sin x \, \mathrm{d}x$$

We split this at 0, but we've already found the integral from 0 to infinity is divergent, so this integral is divergent.

improper integrals

If f is continuous for all x on $(a,\, b]$ and $\lim_{x \to a^+} |f(x)| = \infty$ we define the improper integral with infinite discontinuity at its lower limit

$$\int_a^b f(x)\, \mathrm{d}x := \lim_{t \to a^+} \int_t^b f(x)\, \mathrm{d}x$$

provided this limit exists.

These remaining types of improper integrals are sometimes called type 2 improper integrals.

improper integrals

If f is continuous for all x on $[a,\, b)$ and

$\lim_{x \to b^-} |f(x)| = \infty$ we define the improper integral with infinite discontinuity at its upper limit

$$\int_a^b f(x)\, \mathrm{d}x := \lim_{t \to b^-} \int_a^t f(x)\, \mathrm{d}x$$

provided this limit exists.

example

evaluate

$$\int_0^4 \frac{\mathrm{d}x}{\sqrt{4-x}}$$

Antiderivative is
-2 sqrt(4-x),

taking limit and
evaluating answer is 4.

example

evaluate

$$\int_0^1 \ln(x)\,\mathrm{d}x$$

This is improper since ln is undefined at 0.
Antiderivative is
x ln x - x + C

Must use l'Hôpital to evaluate limit, answer=-1.

This is Stewart's example 7.8.8.

improper integrals

If f is continuous for all x on $[a,\, b]$ except c ($a<b<c$) and if $\lim\limits_{x\to c}|f(x)| = \infty$, we define the improper integral with infinite discontinuity in the interior

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{t\to c^-}\int_a^t f(x)\,\mathrm{d}x + \lim_{s\to c^+}\int_s^b f(x)\,\mathrm{d}x$$

provided this limit exists.

example

evaluate

$$\int_0^{33} (x-1)^{-1/5}\, \mathrm{d}x$$

This is improper since the function is undefined at 1, so we must integrate from 0 to 1 and from 1 to 33.

Answer=75/4.

This is Stewart's 7.8.33.

comparison theorem

Suppose f and g are continuous and suppose $f(x) \geq g(x) \geq 0$ for $x \geq a$.

If $\int_a^\infty f(x)\, \mathrm{d}x$ is convergent, then so is $\int_a^\infty g(x)\, \mathrm{d}x$

If $\int_a^\infty g(x)\, \mathrm{d}x$ is divergent, then so is $\int_a^\infty f(x)\, \mathrm{d}x$

Smaller than convergent is convergent; bigger than divergent is divergent.

Note: The converse does not hold! Functions that are bigger than convergent functions may or may not converge.

example

is the following integral convergent or divergent?

$$\int_1^\infty \frac{\cos^2(x)}{1+x^2}\,\mathrm{d}x$$

Convergent since smaller than 1/(1+x^2) which converges.

This is Stewart's 7.8.49.

coming soon

- read § 8.1
- webwork 5 due friday
- homework 6 due next thursday
- start extra credit project 2, due 23 november @ 6:00 am

Again, this means that you now have an extra week to do extra credit project 2...